

FOR ALL

SAINT LOUIS UNIVERSITY.



SCAN ME

Beautiful Place.

Saint Louis is our home, and we endeavor to make it a better place by creating inclusive communities that can share in the joy of learning and the city's beauty.

ABOUT US

Biannual inclusive mathematics magazine featuring articles and expositions on a wide range of topics related to mathematics and more.



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TABLE OF CONTENTS

01 Welcome to ForAll

02 Faculty Interview Series

03 How to Make a Soccer Ball
By Russell Blyth

04 R&U and a Summer in California
By Kieran Favazza

05 A Remembrance of Guido
By Brody D. Johnson

06 Students' Perspectives: M6M6

07 Symplectic Alternating Algebras
By Layla Sorkatti & Ozlem Ugurlu

08 Welcoming New Faculty

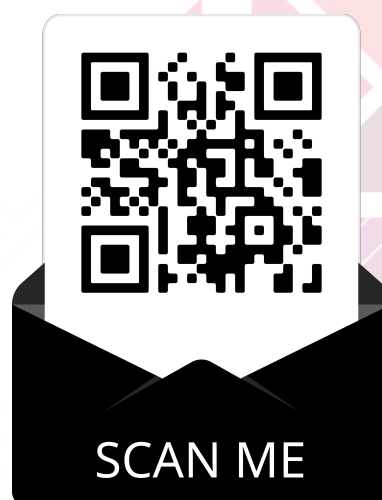
09 Seifert Surfaces and Genera of Knots
By Michael Landry

10 Math Zine
By Lauren Miller

Welcome to *For $\forall$ ll*

We are happy to welcome you all to the second issue of *For $\forall$ ll*, the inclusive mathematical journal of Saint Louis University! We, faculty and students of the Department of Mathematics and Statistics at SLU, love St. Louis - it is our home, and we endeavor to make it a better place by creating inclusive communities that can share in the joy of learning and the city's teeming beauty. When we learn math together, we learn about each other, and when we learn about each other, we strengthen our minds and spirits in ways that transcend the mathematical puzzles at hand. In this way, we can gain a glimpse of understanding of each other - our diverse backgrounds and identities coalesce in our common desire to understand something so enormous and elusive as mathematics. To foster this learning, we have compiled a great selection of mathematical resources, including both informal and formal math articles, interviews with faculty, personal mathematical narratives, and fun puzzles. We thank our faculty and students for their contributions that make this magazine possible. We hope our offerings can bring together all lovers of knowledge - no matter their religious, sociopolitical, racial, ethnic, or mathematical background. This journal is an interdisciplinary one - while its content may be oriented towards mathematics, students, and researchers of all areas can find great elucidation in the many intersections between mathematics, our lives, sociopolitical issues, and research in every field. All who endeavor to persevere in learning deserve to share in its exuberance - help us to provide the framework in which we can all rejoice in our common love. Please enjoy our selection of content.

Follow us and submit your own on our website!



This is a journal of the people in our community, and it is strengthened by any art, puzzles, or articles the community can provide.

For $\forall$ ll Staff and Editorial Board

- Editor in Chief - Özlem Uğurlu, Assistant Professor at Saint Louis University
- Student Editor - Kieran Favazza, Undergraduate student at Saint Louis University

Faculty Interview Series

Interviewing: Stacey Harris

Interviewed by: Kieran Favazza



Introduction

What role can mathematics play in our lives? Of course, the answer to this personal question is one encountered through the experience of doing mathematics, and every person experiences the connection between themselves and mathematics differently. All those who dare to consider mathematics, however, are confronted with a personal and metaphysical dilemma. In doing mathematics, we like to think of ourselves as sleuths of the mystical truths of the universe - uncovering the secret language of reality and expressing its beauty. But how are we doing such a thing? We build understanding through arbitrary definitions - we throw made-up words and symbols around to capture ideas. In this way, we are mere finite approximators of the universe; inventing notions and moving around symbols to make statements that are not necessarily inherent in the physical substance of the universe. How do we reconcile these seemingly contradictory conceptions of math? Is mathematics something created and imaginative, or

is it something scientific and universal? Mathematics is precisely the intersection of the imaginative and the fundamental - it is the human reflection of the divine truth of reality. Thus, mathematics is art; it is that artistic experience of the forces of nature, manifested in the works and concepts we produce. Just as a painting is a natural continuation of the mysterious artistic forces of nature through the human being, a mathematical result is the natural continuation of the mathematical forces of nature through the mathematician. Dr. Stacey Harris, professor in the Saint Louis University Department of Mathematics and Statistics, is one perfectly in the intersection of the imaginative and intellectual realms of nature. She shares how she experiences mathematics and the role it plays in her life, playing on the ways that the vastness of imagination and its ties to intellectual thinking change one's world and mind.

Interview

Dr. Harris, how did you come to be a mathematician?

It all started with the golden age of science fiction - most consider it to be age 12. When I was nine years old, my mother introduced me to science fiction. I read the classics like Asimov and his children's novels and got hooked. From there, I was fascinated by all there was to know about the universe - about science, space, and the imagination. I am very thankful for this exposure - it was an anomaly for a woman, much less a suburban mother like mine, to be interested in science fiction at this time. She later got me a subscription to *Scientific American*, and my interest in science only grew. By the time I got to 10th grade, I had picked up a few books on relativity and spacetime and knew that I wanted to do something with physics, mathematics, and the universe. As I went through my undergraduate studies at Case Western Reserve University, I took just about every math and physics course there was to take, and it came time for me to decide between math and physics. I loved the imaginative and scientific aspects of both, but I ended up choosing mathematics because it was the realm where I felt I could *prove* something and know that it's

true - pure physics is a different type of activity. From there I followed my passions.

What were some of your favorite classes as a student?

I loved differential geometry. I had an old textbook that was notoriously hard to read, and I couldn't make heads or tails of it. However, once I took a differential geometry course, I more deeply understood the objects and the properties that we study. Just like science fiction, it's an expanse of imagination. It was the language of relativity, and it began to make sense to me. In graduate school, I took a seminar in category theory with Saunders Mac Lane each semester I was there. That sort of thing - the universality to it - would later be something I combine with mathematical physics in a very natural way - more on the boundary of spacetime later.

Dr. Harris explains that she couldn't possibly ever use her mathematical brain without using her imaginative one - it's her brain! Her fondness for the imagination of science fiction, her attention to scientific detail, and her interest in the universal and fundamental are the first of our glimpses into the unique experience that is mathematics through Dr. Harris.

Dr. Harris's Research

In the early 1970s, Robert Geroch introduced the notion of causal boundary - a boundary on this object that we might call our spacetime. It later appeared in a book by Stephen Hawking and George Ellis, where it was to be found by Dr. Harris. Most had assumed this causal boundary was something interesting but hard to calculate, but Dr. Harris has calculated it - for several sensible spacetimes - and it has been her work to show that this boundary is something natural and universal using the language of category theory. The structure of this boundary had been erroneously posed and in need of investigation, and Dr. Harris described the structure of the boundary naturally and universally.

What is the causal boundary?

First, let us describe the type of object with which we wish to work. A spacetime is a manifold - a mathematical structure with components describing dimensions of space and one of time such that at each point, the space looks flat if one zooms in far enough - like the world in our daily lives. Points in the spacetime are events - points in space at particular times. If we think about our lives, we are like curves moving through this mathematical object from the past to the future. Our lives are finite, but what about something more persistent - a curve that is directed toward the future and continues indefinitely? We'd like to put an endpoint on the future for these kinds of curves - the end of their causal lives. We create an artificial boundary for their causal lives by looking at the past of a curve. We say that the past of a curve

through our spacetime is the set of all those events that it observes. What does it observe? No massive particle can move faster than the speed of light according to the theories of relativity. If we imagine the kinds of curves that light particles travel through our space-time, the curves that we consider cannot be steeper (faster) than the light curves. Thus, everything inside this space consists of points on the curves that are slower than light curves, and we visualize this space as a cone. If two curves observe the same past, then they should both go to the same future endpoint. If they have different pasts, they have different endpoints. But can we put a richer structure on this idea? If we say something about one endpoint, how does it relate causally to other points in spacetime? If we imagine nice sorts of spacetimes where parallel curves directed toward the future observe the same pasts, then we know the kinds of endpoints that we think about. But what about the curves that run off to infinity in a different way? They could run off to infinity in all sorts of directions, and they could do so at many different times. How do we glue all the endpoints together to have a nice structure? Dr. Harris showed that we can describe this structure like a cone as well - and in a very natural sense. This imagination of the boundary as a cone had been an idea in physics for some time before - an indication that this structure is the right way to think about the boundary of spacetime.

Towards the Current Research

The boundary and its structure that Dr. Harris illuminated were considered for certain well-behaved models of spacetime - but what if we don't make any such nice assumptions about our spacetime? If we just have curves through spacetime - observers like ourselves - what kinds of measurements about the curvature of spacetime would they need to make to conclude that the causal boundary of spacetime is like that of our well-behaved models? As it turns out, Dr. Harris proved that as long as our measurements of curvature don't blow up to infinity too quickly, we are guaranteed a nice boundary.

A Further Generalization

There is a way of thinking about spacetime that is slightly more general than our notion with manifolds. This generalization is more fluid, allowing us to think about more strange physical and time-related phenomena. If instead we consider spaces that are not necessarily manifolds, but just spaces with a notion of distance that we can measure, along with some information about how time separates points (called a Lorentzian length space) how do we say what it means to be "nice" like in our earlier construction? Under what circumstances can we conclude that it has the boundary and structure of this well-behaved sort of space? Dr. Harris is currently working with a team of women in mathematical physics to answer these questions, and the work is proceeding rapidly.

What are some good practices of mathematics?

One should always have examples in mind of the kinds of things they study. Mathematical ideas are born from examples, and understanding them is certainly fundamental to understanding any sort of concept. Whenever I read something, I am always asking: what is your motivation? Why do we want to look at the ideas we have? It is crucial to be able to generalize and recognize the key aspects of examples that we care about.

How can one deal with mathematical frustration?

Take a shower and think about the problem there. Take a break - clear the mind. Take a walk; go bicycling. I can't begin to estimate the number of times I've had a mathematical breakthrough just by walking. During some time I spent at the Perimeter Institute in Waterloo, Canada, I came to many mathematical realizations during chamber music concerts. So it's great to get out of the house, get away from pencil and paper, and just use your head.

It seems that mindfulness is incredibly tied up with mathematics and dealing with mathematical frustration. How does your experience of mathematics influence your mind?

This type of thinking in mathematics is one that I have a particular acumen for - analyzing and generalizing situations, deciding what parts of them are important, and realizing how to communicate them. This influences my life in many ways. In the university political organizations that I have been and continue to be a part of, I use this thinking to communicate and reach solutions for issues that faculty, students, and I care about. But this also applies to turning the lens on myself. I have reorganized my own self-conception in terms of gender on more than one occasion in my life. I think of myself very much as a gender theorist - a large part of it is looking at myself, but another large part of it is attempting to understand society. I consider myself to be in the mold of the logical positivists; I hold that the meaning of a word (like "male" or "female") is observed by how it is used in its context. To see the definition of "gender," "masculine," or "feminine," look at what happens when that word is used. How we understand this helps to understand what happens in the minds of people when we use certain concepts, and this sort of activity is how I see what happens in my mind. It is puzzling how these concepts get developed and used, but as I consider myself, they make more sense to me, and I understand more about myself. I use this mathematical sort of thinking when I consider religion, as well. For instance, I am Jewish in background, and the mourner's kaddish is a ritual that I enjoy considering. It is a traditional prayer in Aramaic given by a person recently grieving a loss. How do I see this and its point in my life? I see it as reaffirming,

in the face of tremendous personal loss, the inherent value in the community of one's fellows and the ideals and values of their tradition. I realize this through this analysis of the context of the words that I hear. I want to know what a person feels inside when we use certain words and concepts. I am fascinated by all the many different ways there are of being human.

How are you human?

Not like anyone I know. Among those in the transgender community, it can be commonly heard that one knew their transgender identity their whole life, but I have conceptualized myself and my gender in different ways at different times. I have done more in-depth examinations of myself tied up with the mathematical, analytic way of thinking. Suppose someone came up to me and said, "What makes you think you're *really* a woman?" I can think of four mutually contradictory responses:

1. Of course I am really a woman. I can argue that very strongly.
2. Of course I am not really a woman. I can argue that strongly too.
3. That is an ill-posed question that does not have a proper answer. I can argue that strongly.
4. Ill-posed or otherwise, it is the wrong question to ask. I can argue this very strongly as well.

But none of these is my answer. Mine, from a feminist view, is that I wish to move through society as an amicable person, be myself, and fit into conceptions as I see fit according to my understanding of those ideas, all the while not giving an inch in the affirmation of my identity. In realizing this, I put on my logical positivist hat and observed how the words of gender affect me. Over some time, I took note of the times I was gendered male versus female. I noticed my emotional response to these situations, and I felt more pleased when gendered female, and more annoyed when I was gendered male. This played into my reconceptualization of my personal gender. The whole idea of gender is fascinating and amusing; I love to analyze it and my emotions relating to it. This way of conceptualizing myself - giving weight to the emotional experience rather than the prescribed "meaning" of a word - is how I want to move through life and society.

Conclusion

Dr. Harris speaks to how mathematics plays into her life and self-conception. The type of thinking that mathematics brings - the intersection of creation and discovery - with all its frustrations and excitements, truly molds one's mind and causes their experience of life to change. Dr. Harris turns her aptitude for imagination and investigation onto herself, using her mathematical

thinking to assess her place in the world and the type of life that she wants to live. She experiences the world through herself and combines her mathematical nature with the universe's seeming mathematical reality to uncover the truth and meaning in her life. Dr. Harris challenges our conception of mathematics and the way it plays into our lives. She continues to push the boundaries of spacetime while pushing the boundaries of our conceptions of imagination, science, truth, and identity.

Billiken Challenge

Challenge 1

Consider a multiplication table of numbers from 1 to n . Shading one or more squares of the table can result in a shaded rectangle:

1	2	3	4
2	4	6	8
3	6	9	12
4	8	12	16

Prove that the number of possible shaded rectangles is equal to the sum of the numbers in the table.

Challenge 2

Consider a square table of n^2 odd numbers, arranged as shown below for $n = 5$:

1	3	5	7	9
11	13	15	17	19
21	23	25	27	29
31	33	35	37	39
41	43	45	47	49

Prove that both diagonals sum to n^3 and the sum of all numbers in the square is n^4 .

Bryan Clair, Ph.D. is an Associate Professor at Saint Louis University. He is interested in topology and combinatorics. We thank him for preparing these challenges for us! These facts were observed by the Uzbek writer Ibn Sina in *The Book of Healing*, published in 1027. For solutions, see our website at <https://sites.google.com/view/forallmath>.

Women Mathematicians Corner

The Association of Women in Mathematics (AWM) started a project called "EvenQuads" to celebrate their fiftieth anniversary by designing a playing deck of cards and honoring notable women mathematicians. Thanks to AWM for permitting us to share these playing cards in our magazine.



Laurie Rubel
(b. 1970s)

Laurie Rubel received her BS in mathematics and computer science from Haverford College (1992), then was a K–12 mathematics teacher while earning her PhD from Columbia University (2002). She was on faculty at Brooklyn College, CUNY (2003–2018) before moving to the University of Haifa. Rubel's research focuses on sociocultural contexts and equity in mathematics education. At Brooklyn College she was active in GLARE (GLBTQ Advocacy in Research and Education).

Rubel was awarded the Young Scholar Award from Knowles Science Teaching Foundation (2006), and an NSF CAREER award (2008–2014). She received a Fulbright Senior Scholar Award (2017) and was an invited speaker for the 2018 annual meetings of AMTE and PME.

This poster is part of the EvenQuads project, which combines innovative mathematical card games with learning about amazing women mathematicians. The project celebrates the 50th anniversary of the Association for Women in Mathematics.

If you want to learn more about EvenQuads, scan the QR code below.

EVENQUADS



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Kavita Ramanan
(b. 1970s)

Ramanan was awarded her applied mathematics PhD in 1998 from Brown University. She is a named professor at Brown University and served as Associate Director at ICERM. Mathematically, Ramanan's interests are in probability theory, stochastic processes, and their applications to operations research, engineering, statistical physics, and neuroscience. She founded a math outreach group at Brown called the Math CoOp and SEAM (Social Equity and Applied Mathematics) seminar. Ramanan has won the INFORMS Erlang Prize (2006), the IMS Medallion (2015), a Guggenheim Fellowship (2020), and the Vannevar Bush Faculty Fellowship (2021). She is a fellow of the IMS (2013), the AMS (2018), INFORMS (2018), the AAAS (2019), SIAM (2020), and the AAA&S (2021).

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EVENQUADS



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How to Make a Soccer Ball

By Russell Blyth

Professor, Department of Mathematics and Statistics, Saint Louis University

1 How not to make a soccer ball

Imagine that you are across the Atlantic Ocean in the United Kingdom, on your way to a soccer game (or in the U.K., a *football* game) when you notice a road sign like the one in Figure 1. On the sign, we see a soccer ball constructed from hexagonal panels. Something doesn't look right. After the game is over, it is still bugging you. Your instincts are correct; the design is impossible for a soccer ball. Yet these signs have been in wide use in the U.K. since 1994.



Figure 1: British road sign for a football (soccer) stadium
(Credit: <https://www.gov.uk/guidance/traffic-sign-images>)

About five years ago, Australian-born, now England resident, stand-up comedian and mathematician Matt Parker petitioned the U.K. government to change the signs to a correct soccer ball design. You can see his description of the problem in a YouTube video¹ or Chapter 4 of his book “Humble Pi.” Parker explains in the video why a soccer ball made only from hexagons is impossible. The U.K. government responded as follows to the petition, which garnered 22,536 signatures: “The Government considers the current football symbol has a clear meaning and is understood by the public. Changing the design to show accurate geometry is not appropriate in this context.” So much for the value of mathematical accuracy!

1.1 Polyhedra

We are going to consider the construction of soccer balls only from regular polygons stitched together (a *regular* polygon has equal length sides and interior angles),

¹<https://www.youtube.com/watch?v=btPqKAGyajM>

although we should note that many currently manufactured soccer balls are made with non-polygonal shapes. Our soccer balls will be inflated versions of *polyhedra*. We can think of a polyhedron as a three-dimensional shape with flat polygonal faces, having straight edges and sharp corners (called *vertices*), such as the icosidodecahedron of Figure 2. Some polyhedra are not suitable models for soccer balls (for example, see Figure 3), so we restrict our attention to *convex* polyhedra, for which line segments between any two interior points lie entirely within the solid.

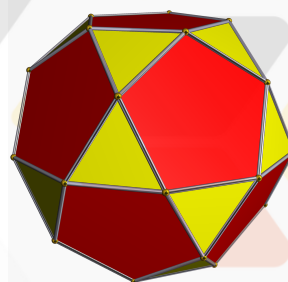


Figure 2: An icosidodecahedron
(Credit: Robert Webb's Stella software
<http://www.software3d.com/Stella.php>)

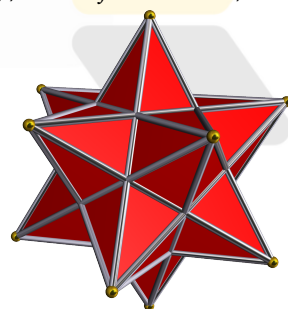


Figure 3: A (non-convex) stellated dodecahedron
(Credit: Robert Webb's Stella software
<http://www.software3d.com/Stella.php>)

Among the collection of convex polyhedra are the five regular Platonic solids, each of which has *identical* regular polygonal faces. For example, a cube is constructed from six squares (see Figure 4).

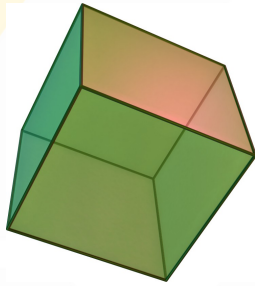


Figure 4: A cube
(Credit: <https://en.wikipedia.org/wiki/Cube>)

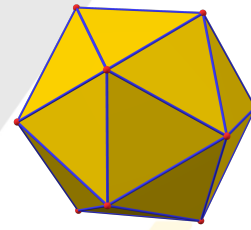


Figure 7: An icosahedron
(Credit: https://en.wikipedia.org/wiki/Regular_icosahedron)

The other regular Platonic solids are the regular tetrahedron (Figure 5), the octahedron (Figure 6), the icosahedron (Figure 7), and the dodecahedron (Figure 8).

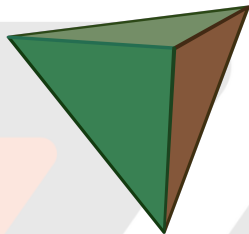


Figure 5: A tetrahedron
(Credit: https://en.wikipedia.org/wiki/Platonic_solid)

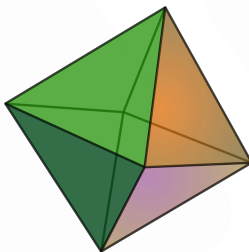


Figure 6: An octahedron
(Credit: https://en.wikipedia.org/wiki/Platonic_solid)

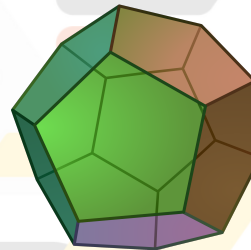


Figure 8: A dodecahedron
(Credit: https://en.wikipedia.org/wiki/Platonic_solid)

where V is the number of vertices the solid has, E is the number of edges the solid has, and F is the number of faces the solid has. For example, a cube (Figure 4) has 8 vertices, 12 edges, and 6 faces; when we calculate the Euler characteristic we get $V - E + F = 8 - 12 + 6 = 2$. You can easily check that the result of the calculation of $V - E + F$ is always 2 for any regular Platonic solid or other convex polyhedron.

The formula extends naturally to any finite planar connected graph (a collection of vertices and edges that can be drawn in the plane without edges crossing where there aren't vertices). The notions of vertices and edges are the same as for polyhedra and the “faces” are now the regions that the plane is divided into by the graph (including the region that lies “outside” of the rest of the graph) – see Figure 9 for an example.

1.2 The Euler Characteristic

The argument Parker uses to show the impossibility of constructing a soccer ball using only hexagons makes use of the Euler characteristic. The Euler Characteristic predates the prodigiously productive mathematician Leonhard Euler by some two centuries; it was first used for regular Platonic solids by the Sicilian mathematician Francesco Maurolico in an unpublished manuscript in 1537. Its use was extended to convex polyhedra by Euler in a 1758 article and has since been generalized extensively. For such polyhedra, the formula is usually expressed as

$$V - E + F = 2,$$

By wrapping the plane around a ball we see that the same formula applies to graphs on a sphere. This means the formula applies to the figures used in the construction of a soccer ball. You might try to prove why the formula $V - E + F = 2$ is valid for all finite planar connected graphs using mathematical induction. Before we move on, we note that the value of the Euler characteristic is different for graphs on other types of surfaces, for example, on a torus (the surface of a typical (one-hole) donut), $V - E + F = 0$ (for a properly subdivided surface of a torus; see, for example, Figure 10). In topology, one learns that the value of the Euler characteristic helps classify what type of surface one is working with. But for our purposes here, $V - E + F = 2$.

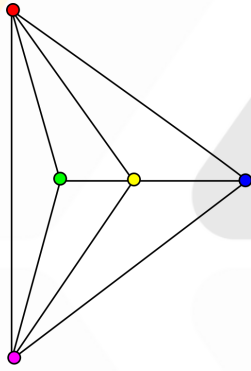


Figure 9: A planar graph with $V = 5$, $E = 9$, $F = 6$
(Credit: https://en.wikipedia.org/wiki/Planar_graph)

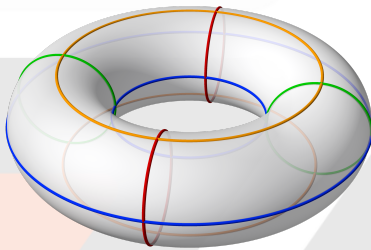


Figure 10: A torus subdivided by circles;
 $V = 16$, $E = 32$, $F = 16$ (Credit: Tilman Piesk,
<https://en.wikipedia.org/wiki/Torus>)

1.3 The impossibility of a sphere tessellated by hexagons

Let's review the proof that Parker discusses in the video that a soccer ball cannot be made just from hexagons. Suppose it could be, using n hexagons. Thus $F = n$. Now count the number of edges. Each hexagon has six edges, which gives $6n$ edges in total – until we stitch the hexagons together to make the ball. We see then that each edge is shared by two hexagons, so we've overcounted the number of edges by a factor of two. Thus $E = 3n$. Finally, we count the number of vertices. Each hexagon has six vertices, so we have a total of $6n$ vertices before stitching. We don't know for sure how many hexagons come together at every vertex, but we do know that there are at least three hexagons around each vertex (as there are in the construction of Figure 1) since just two hexagons around a vertex are impossible. This is enough for our argument – the number of vertices is at most $6n/3 = 2n$, that is, $V \leq 2n$. When we compute the Euler characteristic we get $V - E + F \leq 2n - 3n + n = 0$. But, as we know, on a sphere we must have $V - E + F = 2$, so we have reached a contradiction; we cannot construct a soccer ball only from hexagons.

2 How to actually construct a soccer ball

There are many ways we could construct a soccer ball from regular geometric figures. We could construct a soccer ball using eight equilateral triangles, for example. This design would be an inflated version of an octahedron (Figure 6).

In fact, one could use any of the five regular Platonic solids as the basis for constructing a soccer ball, but it's unlikely that you've ever seen any soccer balls constructed this way. We're used to seeing hexagons used for soccer balls – as in Figure 11. Now we notice something else. The ball has pentagons (the black panels) as well as hexagons (the white panels), with each pentagon surrounded by five hexagons. Each vertex has degree 3 – that is, is surrounded by three panels, two hexagons, and one pentagon. We see the ball constructed, so it must be possible. How many hexagons and pentagons (and how many faces in total), and how many edges and vertices does this construction have? Can a soccer ball be constructed from hexagons and pentagons in other ways? Let's find out, again using the Euler characteristic formula.



Figure 11: A soccer ball, based on a truncated icosahedron
(Credit: [https://en.wikipedia.org/wiki/Ball_\(association_football\)](https://en.wikipedia.org/wiki/Ball_(association_football)))

Let H be the number of hexagons used in constructing a ball, and let P be the number of pentagons used. Then $F = H + P$. We note a relationship between H and P based on the restriction that each pentagon is surrounded by five hexagons, remembering also that each hexagon is surrounded by three pentagons (and three hexagons). Thus $H = \frac{5P}{3}$, since we count each hexagon three times when we multiply the number of pentagons by five. This gives us $F = \frac{5P}{3} + P = \frac{8P}{3}$. Now count the edges. Each edge is counted twice if we count edges as $6H + 5P$, so $E = \frac{6H + 5P}{2} = \frac{6 \cdot \frac{5P}{3} + 5P}{2} = \frac{15P}{2}$. Finally, we count the vertices, which are each counted three times by $6H + 5P$, so $V = \frac{15P}{3} = 5P$. We're ready for the Euler

characteristic formula:

$$V - E + F = 5P - \frac{15P}{2} + \frac{8P}{3} = \frac{30 - 45 + 16}{6}P = \frac{1}{6}P,$$

which must equal 2, so we conclude that we must use $P = 12$ pentagons and $H = \frac{5 \times 12}{3} = 20$ hexagons (32 total faces) to construct a soccer ball using hexagons and pentagons in the manner described above. Such a ball has $V = 60$ vertices and $E = 90$ edges. There is no other design that works. The argument is satisfyingly short, too.

The caption for Figure 11 mentions a truncated icosahedron. This figure is formed from a regular icosahedron (Figure 7) by cutting off each of the 12 vertices of an icosahedron one-third of the way along each edge, creating 12 pentagonal faces and transforming the original 20 triangle faces into regular hexagons. Inflating a truncated icosahedron produces the familiar soccer ball design.

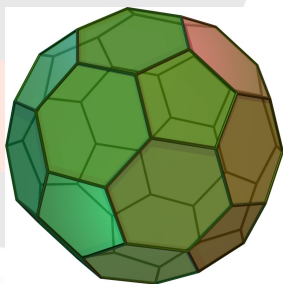


Figure 12: A truncated icosahedron (Credit: https://en.wikipedia.org/wiki/Truncated_icosahedron)

The truncated icosahedron appears elsewhere in nature – it's the underlying shape of the C_{60} Buckminsterfullerene (Buckyball) molecule (see Figure 13). But now we are veering away from the construction of soccer balls, so it's time to blow the final whistle on this article.

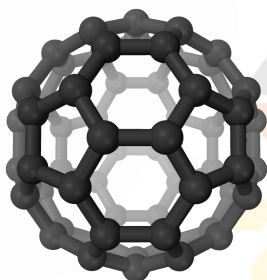


Figure 13: A buckminsterfullerene molecule (Credit: <https://en.wikipedia.org/wiki/Buckminsterfullerene>)

Women Mathematicians Corner

The Association of Women in Mathematics (AWM) started a project called “EvenQuads” to celebrate their fiftieth anniversary by designing a playing deck of cards and honoring notable women mathematicians. Thanks to AWM for permitting us to share these playing cards in our magazine.



Raegan J. Higgins
(b. 1980)

Higgins earned her PhD in mathematics from University of Nebraska-Lincoln (2008). She is currently Associate Professor of Mathematics and Assistant Vice Provost for Faculty Success at Texas Tech University, where her research is in mathematical biology. She has worked throughout her career to improve the training of pre-service teachers and to increase the participation of women and people of color in STEM. She is also co-director of the EDGE Program (2017–) and a co-founder of the *Mathematically Gifted & Black* website. She received AWM's Service Award in 2020 and its Gweneth Humphreys Award in 2021. Her outreach activities have been funded by the NSF Bridges Across Texas Louis Stokes Alliances for Minority Participation grant (2017–2026).

This poster is part of the EvenQuads project, which combines innovative mathematical card games with learning about amazing women mathematicians. The project celebrates the 50th anniversary of the Association for Women in Mathematics.

If you want to learn more about EvenQuads, scan the QR code below.

EVENQUADS



Aisha Nájera
(b. 1981)

Aisha Nájera completed her BSc at the National University of Mexico (2004). She worked for IBM (2008–2010) doing software implementations, and then earned her PhD in applied mathematics at Claremont Graduate University (2015). She is currently a mathematician at the RAND Corporation using mathematical modeling and artificial intelligence in areas including supply chain management, impacts of emerging technologies, and violence against women. Nájera received a Spotlight Award from RAND for her work co-organizing an NSF-funded workshop for Women in Mathematics and Public Policy at IPAM-UCLA (2019) and co-editing the resulting volume *Research in Mathematics and Public Policy* (2020).

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EVENQUADS



REU: A Summer in California

By Kieran Favazza

Undergraduate Student, Department of Mathematics and Statistics, Saint Louis University

Introduction

This past summer, I spent eight weeks at a Research Experience for Undergraduates (REU) at California State University, San Bernardino (CSUSB). I lived with a few peers and spent every day with them thinking and learning about math. We enjoyed the beauty of California in the summertime and were guided by our mentors to expose us to mathematics research - we pushed ourselves to think about math in new ways and tried to balance our math journey with relaxation and exploration of the cities and beaches around us. It was a truly amazing experience that I would love to share with you.

What are REUs?

Research Experiences for Undergraduates (REUs) are programs put on by universities around the country for the purpose of exposing undergraduate students to research - programs exist for physics, biology, engineering, mathematics, and a range of other STEM-related fields. Most REUs last from about five to ten weeks over the summer and provide students with housing and a stipend, but financial details vary from case to case. Often, groups of students work on problems or experiments together, guided by a professional mentor. These are great opportunities to get a taste of what real life is like within research-related fields. If you are thinking about pursuing a career in any sort of research-related field or plan on pursuing graduate studies, an REU is a great way to feel out these possibilities, grow as a student, and build your resume. To find REUs in your field, you can look up “undergraduate research opportunities” in that field or visit the NSF website at <https://www.nsf.gov/crssprgm/reu/>.

My Experience

I was very nervous in the days leading up to my departure for California. I was very scared that I wouldn't be cut out for research or that I wouldn't make friends. But I arrived to find some of the most interesting people I have ever met. Over the course

of the eight weeks that I was there, I became close friends with each of the seven other people in the experience with me. We bonded over our love of mathematics and all the related quirks that come with it, and we immediately fell into patterns of relaxation and research as we went about our days in the math building of CSUSB, having coffee together each morning, playing cards in droves of games throughout the day, and talking to each other about what new idea is making us pull our hair out at the whiteboard all day. I can't express how truly fulfilling it was to sit in a room all day and just think about things that I find interesting (and get paid for it too!). The problems and ideas that I was considering required some background knowledge in a couple of areas of math, but the first week or two of the experience was devoted to giving every student enough background to be able to pursue some interesting questions. After the first few weeks, we were told to just follow our noses and try to think about anything that might interest us. Throughout each day, we would have meetings with our advisor to discuss the types of things we were thinking about. These too were so enlightening and fun - our advisor, Dr. Corey Dunn, was a wonderful person who was helpful, patient, and willing to engage us on any subject, and he genuinely cared about our well-being, always making sure to have fun in whatever we were doing.

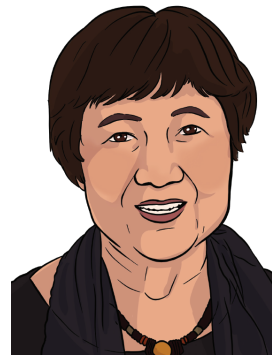
In fact, around the seventh week or so, even though I was loving every second of my time there - the friendships, the scenery, the food included - I found myself frustrated and discouraged. I felt like the work I was doing wasn't enough - that I was the only one not doing something amazing. It felt like the research I had done was elementary and not very interesting, and I was broken up about it. But my advisor sat down with me, with the genuine and friendly calmness that he brought to everything, and listened to my feelings. He related his experiences of imposter syndrome in graduate school and the many times he thought he couldn't cut it. But he assured me that my feelings of inadequacy are natural occurrences in the life of any dedicated mathematician and that my contributions were not meaningless. He asked me questions about my research and pushed the conversation to a point

where I could realize some value in the work I had done. I am forever grateful for these moments with him, and I think this experience speaks to a universal feeling within mathematics. We constantly feel like we are the only ones misunderstanding, solitary in our frustration that we aren't achieving or grasping something important. But all it takes is the presence or the comforting encouragement of one friend or teacher to pull us out. We are constantly reminded of our connection to the mathematical community around us - that mathematics is not something meant to be undertaken alone.

And I certainly shared the joy of doing mathematics with others this summer. By day, my peers and I worked on ideas and talked through math with each other, and by night, we cooked together, ate together, made math jokes together, and talked into the night about our lives, inevitably coming back to how math influences us. We tried our best to create time away from math so as to not burn ourselves out (we had a "no math after 9 PM" rule), which succeeded in balancing our intense mathematical lives with the exuberant and beautiful experience of being near Los Angeles and Southern California beaches all summer. I will never forget the moments I shared with my friends in such a beautiful place and the immense impact they had on the way I think about math. I will be forever grateful for the mathematical exposure and experience I gained, and I can truly say that I was transformed as a person and as a mathematician this summer. If you want to make friends that will transform your thinking and be mentored to grow substantially in your field, an REU is a fantastic way to get started!

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Chuu-Lian Terng
(b. 1949)

Chuu-Lian Terng received her PhD from Brandeis University in 1976. She spent most of her career at Northeastern University and the University of California at Irvine. Terng is the author of over 60 research papers in submanifold geometry and integrable PDEs. Her co-authored book *Critical Point Theory and Submanifold Geometry* had 38 editions between 1987 and 2006 and was published in three languages. She has been active in the AWM throughout her career and served as AWM president (1995–1997). Terng has been on several editorial boards, including for the *Proceedings of the AMS*. Her numerous recognitions include a Sloan Fellowship (1980), a Humboldt Senior Scientist Award (1996), and an invited ICM talk (2006).

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EVENQUADS



Raman Parimala
(b. 1948)

Raman Parimala was born in India and earned her PhD from Bombay University (1976). She is currently a Distinguished Professor at Emory University. Her research is in quadratic forms, Galois cohomology, and algebraic groups. She has published over 100 papers and advised 10 PhD theses. She was an invited speaker at the International Congress of Mathematicians (ICM) in 1994 and 2010 and gave the AWM-AMS Emmy Noether Lecture at the Joint Mathematics Meetings in 2013. Parimala is a fellow of all three Indian academies of science and of the AMS (2013). In 2020 she became one of 11 Indian female scientists, and the only mathematician, to have a chair at an Indian institution named in their honor.

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EVENQUADS



A Remembrance of Guido

By Brody Dylan Johnson

Associate Professor, Department of Mathematics and Statistics, Saint Louis University

In the fall of 1998, as a second-year doctoral student at Washington University in St. Louis, I had the good fortune to enroll in Measure Theory with Professor Guido L. Weiss. Now, some twenty-five years later, I find myself teaching Measure Theory and reflecting not only on the course I received that semester, but also on the mathematician, mentor, and friend from whom I learned the subject. Guido (as he preferred to be called) injected a lot of his personality into his teaching, and as I review my handwritten notes from the course, I can hear his voice and almost put myself back in that time and place. Despite this level of familiarity, I also rediscovered some forgotten gems in the notes that I may not have fully appreciated as a student.



Figure 1: Professor Guido L. Weiss in 2002.

Guido, with his Shih Tzu, Thor, in tow, started the course at a brisk pace, frequently delegating the proofs of elementary results to the students. The first class opened with a discussion of the Banach-Tarski Paradox that motivated the notion of a σ -algebra in the development of measures. The second class was largely occupied by a study of σ -algebras, including several realizations of the Borel σ -algebra on $\mathbb{R}$, and ended with the formal definition of a measure. Early in the next class, Guido interrupted the flow of mathematical development to share a bit of wisdom.

Weiss' First Principle: I don't know much, but I'll use the only things I know to prove the result.

Guido then proceeded to illustrate this principle by deriving the key properties of measures, e.g., monotonicity and countable subadditivity, from the few concepts that had been introduced in the course. The fourth class bore witness to the introduction of *Guido's Principle of Backwards Induction*, which was used to investigate the countable disjoint additivity of outer measures in a proof of Caratheodory's extension theorem for outer measures. Guido had a flair for emphasizing ideas that were more fundamental than the proof where they first appeared and used these philosophical diversions both to entertain and to enlighten.

A week or two later, Guido embarked on the development of Borel measures associated with increasing, right-continuous functions. In the middle of one proof, he paused to present another important piece of sage advice.

Weiss' Seventeenth Principle: If you've not used all of the hypotheses in a proof, then either the statement of the theorem is incorrect or you have not proved it yet.

The continuation of the proof managed to involve the previously unused hypotheses and all was well. The relatively large number had me wondering about the numbering scheme of Weiss' principles, which I'm sure Guido could have explained if anyone was brash enough to ask. In certain instances, no number was given, as was the case just before Guido stated a characterization theorem for Borel measures on $\mathbb{R}$. This principle appeared in my notes as follows.

Another of Weiss' Principles: Natural things are natural.

This principle seemed chiefly to suggest that some proofs need not be written out in full detail. In this instance, Guido summarized the proof by simply writing $\mu((-\infty, x]) = F(x)$.

Once the discussion of measures was complete (no pun intended), the course shifted focus to the

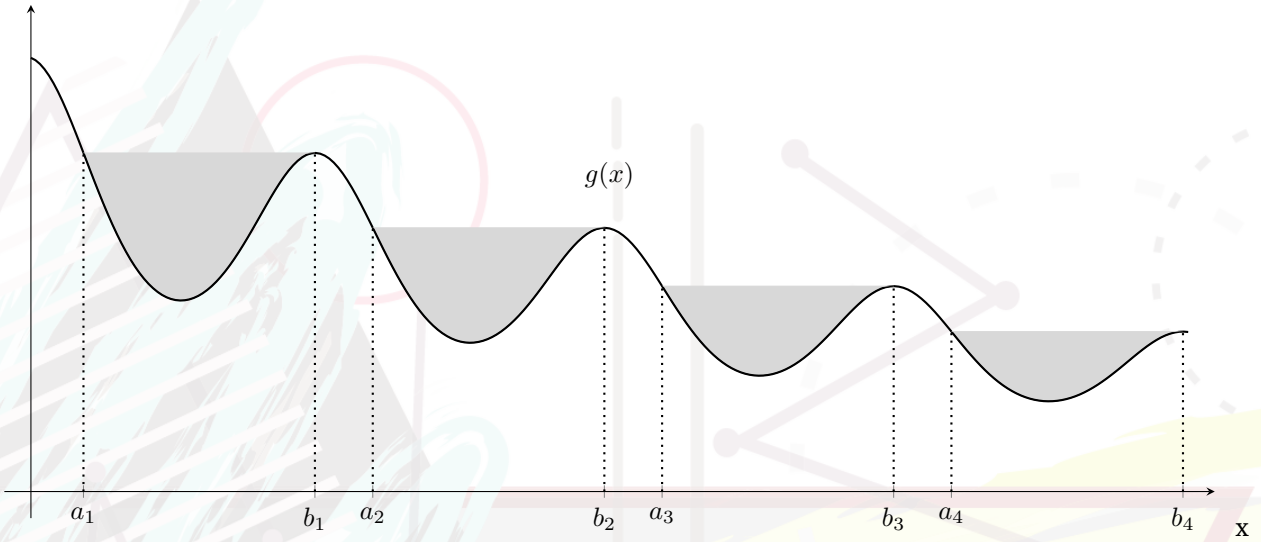


Figure 2: Illustration of the Sunrise in the Mountains Lemma.

measurability of functions. Guido's explanation of the notion of preimage was so elegant that its beauty was almost certainly lost on me at the time: "Given $f : X \rightarrow Y$, f^{-1} is a wonderful function from $\mathcal{P}(Y)$ to $\mathcal{P}(X)$." He followed this fanciful statement by illustrating how the preimage f^{-1} respects complements, unions, and intersections, leading to the conclusion that f^{-1} maps σ -algebras of Y to σ -algebras of X . I've explained this fact many times in my own measure theory courses, but never with the elegance carried through his booming, yet gentle, tone in that lesson.

One of my favorite lessons came in mid-October when the focus of the class shifted from integration to differentiation. Guido broke from the script provided by our textbook and introduced a unit as a classical background for the decomposition of measures, stating a remarkably colorful lemma. Referring to the simple illustration of the lemma given in Figure 2, the basic idea is that as the Sun rises, light shines westward, parallel to the ground, and each mountain peak casts a shadow on the next peak to the west. The lemma provides a nice description of the set, E , consisting of those points within (a, b) in the regions of shadow.

Lemma (Sunrise in the Mountains Lemma). Suppose g is a continuous function on the closed and bounded interval $[a, b]$. Let

$$E = \{x \in (a, b) : \exists \xi \in (x, b] \text{ s.t. } g(x) < g(\xi)\}.$$

Then E is an open set (possibly empty) and, hence, can be written as a union of at most countably many disjoint open intervals,

$$E = \bigcup_k (a_k, b_k),$$

with $g(a_k) \leq g(b_k)$ for each k .

Proof. Assume that $x \in E$. Then, there exists $\xi \in (x, b]$ such that $g(x) < g(\xi)$. Choose $\varepsilon = g(\xi) - g(x)$. The

continuity of g at x implies that there exists $\delta > 0$ such that $|y - x| < \delta$ guarantees that $|g(y) - g(x)| < \varepsilon$. It follows that if $y \in (x - \delta, x + \delta)$, then $g(y) < g(\xi)$ and, moreover, that $\xi \in (y, b]$. In other words, $(x - \delta, x + \delta) \subseteq E$ and thus E is open. As such, E can be written as the union of at most countably many disjoint open intervals (a_k, b_k) .

It remains to prove that $g(a_k) \leq g(b_k)$ for each k . Fix k and choose $x \in (a_k, b_k) \subseteq E$. Then there exists $\xi \in (x, b]$ such that $g(\xi) > g(x)$. Observe that $b_k \notin E$, which implies that either $g(b_k) \geq g(\xi) > g(x)$ or $\xi \in (x, b_k)$. It will be shown that the inequality $g(b_k) \geq g(x)$ also holds in the latter case. Assuming that $\xi \in (x, b_k)$, define x' by

$$x' = \sup \{y \in (x, b_k) : g(y) \geq g(x)\},$$

which is well-defined since $\xi \in (x, b_k)$ with $g(\xi) \geq g(x)$. Now, choose a sequence $\{x_n\}_{n=1}^{\infty}$ in $\{y \in (x, b_k) : g(y) \geq g(x)\}$ that converges to x' and observe that the continuity of g guarantees that $g(x') \geq g(x)$. If $x' = b_k$, then, again, $g(b_k) \geq g(x)$. Otherwise, $x' \neq b_k$, which forces $x' \in (x, b_k) \subseteq E$. Hence there exists $\xi' \in (x', b]$ such that $g(\xi') > g(x')$. If $\xi' \in (x', b_k)$, then $\xi' > x'$ with $g(\xi') > g(x)$, contradicting the supremacy of x' . If $\xi' = b_k$, then $g(b_k) > g(x') \geq g(x)$ once again. Finally, if $\xi' > b_k$, the fact that $b_k \notin E$ forces $g(b_k) \geq g(\xi') > g(x') \geq g(x)$. This completes the argument that $g(b_k) \geq g(x)$ for all $x \in (a_k, b_k)$. The continuity of g now guarantees that $g(a_k) \leq g(b_k)$, completing the proof of the lemma. $\square$

Guido used the Sunrise Lemma to prove that continuous, monotone functions are differentiable almost everywhere and assigned a more general form of the Sunrise Lemma as an out-of-class project. Although we were not proving anything new, this assignment had the feel of real mathematical research, and I enjoyed it immensely.

Guido turned seventy shortly after the Measure Theory class ended and in the next spring, played on our intramural softball team, the *Szegő Colonels*. I organized the team and was a little hesitant to put Guido on the field, despite the fact that he was a lifelong athlete. But Guido was never a bench player. So he played third base for a few innings and recorded a single in his one trip to the plate. He also had a reputation for giving lessons on tennis courts around town, and I tagged along on one occasion to capture a few pictures (see Figure 3).



Figure 3: Guido making a backhand volley on the Christian Brothers College High School Tennis Courts.

Near the end of the academic year, I approached Guido to ask him about serving as my doctoral advisor, and he immediately agreed. I spent some of that first summer reading portions of Guido's famous book with Elias M. Stein, *Introduction to Fourier Analysis on Euclidean Spaces*, before graduating to *A First Course on Wavelets*, which Guido wrote with his former student Eugenio Hernández. The next fall I began attending the long-running Wavelet Seminar, which brought me into contact with many future mentors and friends, such as Marcin Bownik, Eugenio Hernández, Richard Laugesen, Hrvoje Šikić, Darrin Speegle, and Edward Wilson, among many others. Ultimately, I completed my dissertation under Guido's tutelage in May 2002, something that we documented with a couple of pictures in Guido's office (see Figure 4). His office featured a full-sized couch in front of a long blackboard and served as the focal point for wavelet research in St. Louis for many years.

I learned many things from Guido, both big and small. First, an example of the small. Guido was well-known for his exceptional board work, and when underlining a string of words on the board, he would interrupt the line whenever it encountered a *descender* (the typographical term for the part of a g, j, p, q, y, etc. that extends below the line). For whatever reason, I found that so aesthetically pleasing that I adopted the practice in my own handwriting, whether on paper, on a tablet, or on a board. A big thing that I learned from Guido is that seemingly well-studied and well-understood mathematical concepts can



Figure 4: Guido and I posing with my doctoral degree in his office.

often reveal new mathematics when thoughtfully re-examined. This philosophy is reflected in Guido's many contributions to the theory of wavelets and has also influenced the direction of my own research. What I remember most about Guido, however, is that he was always looking out for others and was generous beyond all expectations with his time. He could (and often did) have a million and one things going on, yet when I would meet with him, it was as if he had all the time in the world.

I hope that the reader will not mind me indulging in one more sample from Weiss' Principles, shared by Guido partway through his survey of the interpolation of operators. I *think* I know what message he was trying to impart, but I'll leave it as an exercise for you to interpret it for yourself.

Weiss' Fourth Principle:

Person 1: What is blue, hangs from a ceiling, and whistles?

Person 2: A yellow bird.

Person 1: It's not blue.

Person 2: So, paint it blue.

Person 1: It's not hanging from a ceiling.

Person 2: So, hang it from a ceiling.

Person 1: It's not whistling.

Person 2: That's irrelevant.

Students' Perspectives:

Missouri Collegiate Mathematics Competition

What is the Missouri Collegiate Mathematics Competition?

The Missouri Collegiate Mathematics Competition (MCMC) invites teams of one to three undergraduate students to complete a total of ten challenging mathematical problems over a pair of two-and-a-half-hour sessions. Some problems boil down to clever applications of calculus, while others test students' breadth of knowledge in areas like differential equations, linear algebra, probability, or other advanced topics. The competition is held in conjunction with the Missouri Section meeting of the Mathematical Association of America during the spring semester. Interested students can access problems and solutions from past competitions through the official website <http://sections.maa.org/missouri/contest.html>.

Brody Johnson: A Note from the Faculty Mentor

I took over as faculty mentor for the MCMC team in 2015 while Bryan Clair was on sabbatical. Initially, I was a little intimidated to take on the role because my prior experience with math contests amounted to a single appearance on the Equations team for my middle school (maybe high school?) back in the late 1980s. I like to remind students that, even as a faculty member, I still find many of the problems to be challenging, although I can also attest to the fact that regular practice tends to sharpen one's skill for these types of problems. Sometimes new students will look at the problems from a past competition and get discouraged when they can't solve many of the problems – that's kind of like looking at the final exam for a course before registering. The most successful teams I have seen were made up of students who got past that initial discouragement and started digging into old problems, soaking up ideas, and making connections between things they've learned and strategies for the solution of the problems.

Anna Goranski

If you want to challenge yourself on your knowledge of mathematics, join the Missouri Collegiate Mathematics Competition (MCMC) team. The MCMC is composed of two sessions within the span of two days. Each team can have a maximum of three competitors, and any university or college in Missouri is welcome to register.

I attended the competition in my last semester of senior year and am very grateful I did so. Personally, I had heightened nerves even considering joining the team. In this type of event, it is very helpful to have a general foundation of different topics in math, such as calculus, probability, and linear algebra, of course along with many others. In my experience, I had a gap year in my math studies, which caused a harder time for me to recall my previous knowledge. However, becoming a part of the team reinforced the math I already had learned, and it taught me a variety of new math. It was an incredible opportunity to not only test your own knowledge but also discover other ways to apply the math you already know.

I was lucky to have the teams that I did when I attended the competition. Everyone at the practice was more than willing to help others understand the practice problems we completed. The questions are meant to be difficult (you have two and a half hours to answer five questions in each session), and you tend to learn a lot from your peers. When it was time for the competition, we traveled to Missouri State University in a mini-van from SLU with a total of six students and one coach ... safe to say it was a packed car. However, the drive went by quickly. Before I knew it, we were at the hotel getting ready to go and compete at MSU. The questions at the competition were daunting at first; however, with some guidance and ideas from my other team members, we started to complete them! When the time was up, we were given a solutions sheet to estimate our earned points. The same process occurred the next day. After that competition, the last day of the trip is dedicated to talks and presentations from math professors and faculty across all universities in Missouri. Those talks were filled with rich information, and many were fascinating to listen to due to interactivity with the audience. I am delighted with my experience of preparing and attending the MCMC and would do it again if given the chance.

Ben Schneiderheinze

I have always greatly enjoyed math competitions, so I was really excited to represent Saint Louis University at the MCMC. My favorite part about the competition itself was the team aspect. Mathematics is often seen as an individual endeavor, but working on problems as a team forces you to understand how your skill set fits

in with the skill sets of other members of your team. This brought many of my mathematical weaknesses to light and encouraged me to improve upon them. Outside of the competition, the talks during the day and the banquet at night were great opportunities to learn about and discuss different math topics with professors and students from all across the state.

Kieran Favazza

My experience at the MCMC was amazing and one that I hope to repeat. It is really fun to spend time with people who share your interests, so being in a car or a room with other math students and a beloved math professor was always entertaining and fun for me. We constantly joked and talked about math, increasing our skills, but we certainly took time away from math to just enjoy ourselves on this mini-trip. We ate at many good restaurants along the way, talked and joked with each other (trying to fight robots when we could), and spent a wildly entertaining few hours in the study room of our hotel learning to play Dungeons and Dragons alongside more of our beloved professors.

The competition itself was an invigorating challenge that increased my mathematical skills and practices, and it went by in the blink of an eye. Suddenly, we were discussing all the solutions we almost got or didn't get, and we were on our way to another fun lunch or series of talks at the Missouri Sectional MAA meetings. These too were very enlightening and entertaining, and by the time the talks were over, it was time to announce the winners of the competition and have dinner. Participating in this competition was a great mathematical practice not only for its contribution to my math skills but also for its lessons in dealing with mathematical frustration - I had some good practice in forgiving myself for not figuring something out and being calm under pressure. Overall, the experience was fun and enlightening from the start to the very end.

What is Happening in Ritter?

Association for Women in Mathematics Student Chapter

The SLU Student Chapter of AWM mentors and encourages women and other minorities as they prepare for careers in mathematical sciences. We encourage conversation in the mathematics community among women and other gender minorities. This includes connecting undergraduate students, graduate students, and faculty at both formal talks and casual mathematics gatherings. While this is a math chapter, it is not only aimed at math students but rather at all neighboring fields and interests. We meet on Mondays at 4:00 pm. Check out the events and more information at the [department's website](#) or through [SLU Groups page](#).



Math Club

The Math/CS club meets on Wednesdays at 4 pm in the Ritter Hall Lobby. Events include outside speakers, games, competitions, field trips, and other opportunities to socialize with fellow students. Undergraduate students, graduate students, and faculty are all welcome to attend club events. Find more information about the club on the [SLU Groups page](#).



Colloquium

The department runs weekly colloquia featuring speakers from both SLU and other top national institutions. The events are categorized into researched-focused and education-focused talks. The departmental colloquium meets on Fridays at 4:00 pm with refreshments beforehand in the Ritter Hall Lobby. Check out [department's events website](#) to see the calendar for all departmental events.



Symplectic Alternating Algebras

Layla Sorkatti¹ and Özlem Uğurlu²

[1] Research Scholar, School of Mathematical and Statistical Sciences, Southern Illinois University

[2] Assistant Professor, Department of Mathematics and Statistics, Saint Louis University

From the historical point of view, the word “algebra” means balancing and reduction. It originated from the Persian mathematician Muḥammad ibn Mūsā al-Khwārizmī’s work on the subject known as Al-Jabr around 830 CE, and his work focused on solving linear and quadratic equations. As more mathematical objects were invented over the centuries, modern algebra began to focus on sets of these objects endowed with one or more mathematical operations, such as addition and multiplication. More precisely, a set equipped with a collection of operations satisfying some mathematical identities (i.e., axioms) is called an algebraic structure, and we study these structures in algebra. There are varieties of algebraic structures, such as groups, rings, fields, and algebras. A vector space is another example of an algebraic structure, but what do we mean by a vector space? Consider a line segment with an arrow that indicates the direction of the line; this object is called a *vector* in mathematics. Now, imagine these vectors can be added together, multiplied by scalars, and respect certain mathematical rules. For example, adding two vectors first and then multiplying by a scalar is the same as multiplying each vector by a scalar and then adding them - this rule is known as *distributivity*. Simply put, the collection of vectors on which certain operations can be performed is called a *vector space*. It follows from the definition that algebra focuses on algebraic properties of the addition and scalar multiplication of vectors.

Vector spaces are great tools for analyzing systems of linear equations, and for this reason, this concept is covered in any undergraduate-level linear algebra class. One of the most fascinating features is that every vector space can be generated by a number of vectors. The set of such vectors is called a *basis*. The basis vectors are not dependent on each other, and any vector in the space can be written as a linear combination of these basis vectors. For instance, the basis vectors for the three-dimensional vector space $\mathbb{R}^3$ are $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

Other than algebraic features of $\mathbb{R}^3$, we also have

geometric features involving the length of a vector or the angle formed by vectors in $\mathbb{R}^3$. This geometric intuition can be carried to any vector space if we add an additional structure, called an inner product. Let u, v , and w be vectors in a vector space V and k be a scalar in a field F (for instance, $\mathbb{R}$ or $\mathbb{C}$). An *inner product* is a function $\langle \cdot, \cdot \rangle : V \times V \rightarrow F$ which has the following properties:

1. (Positive-Definiteness) $\langle u, v \rangle \geq 0$ and $\langle u, u \rangle = 0$ if and only if $u = 0$;
2. (Conjugate-Symmetry) $\langle u, v \rangle = \overline{\langle v, u \rangle}$;
3. (Linearity in first component) $\langle ku + kv, w \rangle = k\langle u, w \rangle + k\langle v, w \rangle$.

A vector space equipped with an inner product is called an *inner product space*. As the conjugate of a real number is itself, the “conjugate-symmetry” condition becomes the “symmetry” condition for an inner product on a real vector space. It is also worth mentioning that an inner product on a real vector space is a bilinear form - it is linear in each component separately. However, an inner product on a complex vector space is not, since it is conjugate-linear in the second component. Geometrically speaking, the inner product gives us a way to measure one-dimensional notions; the lengths of vectors and angles between two vectors. Then it is natural to wonder whether there is a way of measuring two-dimensional areas in space. This can be addressed by equipping the vector space with a symplectic form instead of an inner product.

A *symplectic vector space* is a vector space V over a field F equipped with a *symplectic form* that is a function $(\cdot, \cdot) : V \times V \rightarrow F$ satisfying the following properties:

1. Bilinear;
2. (Alternating) $(u, u) = 0$;
3. (Non-degenerate) $(u, v) = 0$ for all $v \in V$ implies that $u = 0$.

Even though the word “symplectic” was first introduced by German mathematician and theoretical physicist Hermann Weyl in 1939, symplectic space was discovered by William Rowan Hamilton while

studying dynamical systems in the 1800s. So, it is safe to say that physics found a geometric structure for mathematics to study.

There are many mathematical properties that a symplectic vector space carries, for example, every symplectic vector space is even-dimensional. Moreover, any symplectic vector space of dimension $2n$ splits as the direct sum of two isotropic subspaces of dimension n , meaning that the symplectic form vanishes on these subspaces. To be more precise, a subspace W of a symplectic vector space V is called an *isotropic subspace* if $(u, v) = 0$ for all u and v from W . Any one-dimensional subspace, Fu , is an example of an isotropic subspace. Further,

$$W \subset W^\perp = \{v \text{ in } V \mid (v, w) = 0 \text{ for all } w \text{ in } W\}.$$

What is a Symplectic Alternating Algebra?

Now, if we add another operation, *binary alternating product*, to a symplectic vector space that interacts with the non-degenerate form in a certain way, we get what is called Symplectic Alternating Algebra. Our goal in this expository article is to define such algebra, study some of its features, and give some examples.

A *Symplectic Alternating Algebra (SAA)* over a field F is a symplectic vector space V over F equipped with a binary alternating product “ $\cdot$ ”: $V \times V \rightarrow V$ such that the law

$$(u \cdot v, w) = (v \cdot w, u),$$

holds for all u, v , and w in V . We refer to this law as *SAA Property*. Notice that

$$(u \cdot v, w) = (v \cdot w, u) = -(w \cdot v, u) = (u, w \cdot v),$$

and this is known as the self-adjointness property. So, the multiplication from the right by v is *self-adjoint with respect to the alternating form*.

As V is an even-dimensional space, we can pick a basis $x_1, y_1, \dots, x_n, y_n$ with the property that

$$(x_i, x_j) = (y_i, y_j) = 0 \text{ and}$$

$$(x_i, y_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

for $1 \leq i \leq j \leq n$. We refer to a basis of this type as a *standard basis*. Suppose we have any basis $u_1, \dots, u_{2n}$ for V . The structure of V is then determined from

$$(u_i \cdot u_j, u_k) = \gamma_{ijk}$$

for $1 \leq i < j < k \leq 2n$ and γ_{ijk} is a scalar from the field F . We refer to such data as a *presentation* for

V . Alternatively, we can describe V as follows: if we take the two isotropic subspaces, $Fx_1 + \dots + Fx_n$ and $Fy_1 + \dots + Fy_n$ concerning a given standard basis, then it suffices to write down only the products

$$x_i \cdot x_j, y_i \cdot y_j, \text{ for } 1 \leq i < j \leq n.$$

The reason for this is that having determined these products, we have determined all the triples $(u_i \cdot u_j, u_k)$ where $1 \leq i < j < k \leq 2n$ since two of those are either some x_i, x_j or some y_i, y_j , in which case the triple is determined from $x_i \cdot x_j$ or $y_i \cdot y_j$. Since

$$(x_i \cdot x_j, x_k) = (x_j \cdot x_k, x_i) = (x_k \cdot x_i, x_j)$$

and

$$(y_i \cdot y_j, y_k) = (y_j \cdot y_k, y_i) = (y_k \cdot y_i, y_j),$$

this puts some more conditions on the products $x_i \cdot x_j$ and $y_i \cdot y_j$.

Some Properties of SAAs

Let V be a symplectic alternating algebra and let u, v be in V . Now, consider the subspace generated by the element $u, uv, uvv, \dots$ in V , (omit “ $\cdot$ ” here), and hence any random element from this subspace is in the form of uv^r for some positive integer r . Combining this with the SAA property, we have $(uv^r, uv^s) = 0$ for some positive integers r and s . Indeed,

$$(uv^r, uv^s) = (uv^k, uv^k) = 0, \quad \text{if } r + s = 2k,$$

$$(uv^r, uv^s) = (uv^k \cdot v, uv^k) = (uv^k \cdot uv^k, v) = (0, v) = 0,$$

if $r + s = 2k + 1$. This implies that the subspace generated by $u, uv, uvv, \dots$ is isotropic and leads us to the following lemma.

Lemma 1. *Let V be a symplectic alternating algebras. If $u, v \in V$, then the subspace generated by $u, uv, uvv, \dots$ is isotropic.*

Next, we focus on the lower and upper central series and provide definitions similar to the ones given in group theory. The lower central series is defined recursively by

$$V^1 = V \quad \& \quad V^{n+1} = V^n \cdot V,$$

and the upper central series by

$$Z_0(V) = \{0\} \quad \& \quad Z_{n+1}(V) = \{x \text{ in } V \mid x \cdot V \text{ in } Z_n(V)\}.$$

We have the following relation between the upper and the lower central series:

Lemma 2 ([7]).

$$Z_i(V) = (V^{i+1})^\perp.$$

Proof. We have $a \in Z_i(V) \Leftrightarrow a \underbrace{V \cdots V}_{i-\text{many}} = 0 \Leftrightarrow 0 = (a \underbrace{V \cdots V}_{i-\text{many}}, V) = (a, V^{i+1}) \Leftrightarrow a \in (V^{i+1})^\perp$. $\square$

As a consequence, we see that $\dim(Z_i(V)) + \dim(V^{i+1}) = \dim(V) = 2n$. Moreover, the upper and lower central series have another special relation:

Lemma 3.

$$Z_i(V) \cdot V^i = \{0\}.$$

Recall that in group theory, we do not have that $G/Z(G)$ is cyclic. Likewise, for any alternating algebra A , $A/Z(A)$ can not be 1-dimensional. It follows that we have the following interesting result

Theorem 4.

$$\dim(V^i) \neq 1.$$

Examples

It is known that the only symplectic alternating algebra of dimension 2 is the abelian one - meaning that $u \cdot v = 0$ for all u, v in V .

In algebra, one of our main goals is to classify algebraic systems. When systems have the same algebraic structure we treat them “as same” even though they are collections of entirely different objects. This notion is known as *isomorphism*. It would be a good exercise for those interested in the subject to show that, up to isomorphism, there are two symplectic alternating algebras of dimension 4: one is abelian, whereas the other one has the following multiplication table:

$$\begin{array}{ll} x_1 \cdot y_1 = x_2 & x_1 \cdot x_2 = 0 \\ y_1 \cdot y_2 = -y_1 & x_2 \cdot y_1 = 0 \\ x_1 \cdot y_2 = -x_1 & x_2 \cdot y_2 = 0 \end{array}$$

The presentation is thus $(x_1 \cdot y_1, y_2) = 1$. Note also that the center of the non-abelian SAAs of dimension 4 is $Z(V) = \langle x_2 \rangle$.

The symplectic alternating algebras of dimension 6 have been classified in [7]; when the field has three elements, there are 31 such algebras, of which 15 are simple.

As a final remark, it is worth mentioning that symplectic alternating algebras originate from a study on 2-Engel groups, and there is a one-to-one correspondence between a certain rich class of powerful 2-Engel 3-groups of exponent 27 and SAAs over the finite field with 3 elements $F_3 = \{0, 1, -1\}$. As SAA is considerably a new type of algebraic structure, many open problems still need to be explored. More information and further development for those interested in learning more about the subject can be

found in [2], [3], [4], [5], [6], and [7].

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Welcoming New Faculty

The department welcomes those who joined our faculty in August!

Dorsa Ghoreishi



Dorsa Ghoreishi joined our department in August 2020 as a post-doctoral fellow, and after three years, she started her new position as an Assistant Professor. Dr. Ghoreishi was born and raised in Iran and moved to the United States to start her doctorate program. She received her Ph.D. in Mathematics at the University of Missouri-Columbia in 2020. Her research is in the field of Frame Theory with a focus on Phase Retrieval. During her time at SLU, she initiated the re-establishment of the Association for Women in Mathematics Student Chapter and received an NSF grant to support her research. She is a Missouri Project NExT fellow and a member of the national organizations of AMS and AWM.

In her free time, she enjoys reading books, rock climbing, running, and spending time with her tortoiseshell cat.

Cody Gilbert

Cody Gilbert joined our department this year as a postdoctoral fellow. Dr. Gilbert obtained his Ph.D. in Mathematics in 2023 at the University of Iowa under the guidance of Ryan Kinser. His thesis involved classifying isomorphism classes of finite dimensional representations of tame algebras using moduli spaces. Currently, he is working on understanding quasi-cluster algebras with a few friends at the University of Minnesota as well as Leavitt path algebras with Ashish Srivastava, here in Saint Louis!

Outside of mathematics, he enjoys writing/playing music on piano and synthesizer as well as charcoal drawing and literature.



Michael Landry



Figure 1: Archives of the Mathematisches Forschungsinstitut Oberwolfach.

Michael Landry joined our department this year as an Assistant Professor after four years as a postdoc across town at Washington University in Saint Louis. Dr. Landry grew up in Santa Cruz, California, and lived in the San Francisco Bay Area and New England before arriving in St. Louis. His work involves low-dimensional topology and dynamics.

He enjoys drawing pictures, doing crossword puzzles, and riding bicycles. It would not be unusual to see him enjoying a meal at Lona's Lil Eats in Fox Park, or quaffing a beverage at the Tick Tock Tavern with friends. He lives in St. Louis' Tower Grove East neighborhood with his spouse, two cats, and many plants. He thinks most effectively when lying on the ground and staring at the ceiling.

Yan Lyansky

Yan Lyansky joined our department this year as a Non-Tenure Track Assistant Professor. Yan was born in Kyiv, Ukraine. He went to Temple University for both undergraduate and graduate studies. His work has been in Probability Theory, Statistical Mechanics, Mathematical Finance, and Mathematical Art. He has taught all over the world, including the Bahamas, Virgin Islands, and Hawaii.

Yan ran two marathons, competed in many Olympic triathlons, boxed with professionals (trained by Joe Black, former number three middleweight in the world), biked across the country twice, and designed Downtube folding bikes for over two decades. He is a two-time Bahamas National Chess Champion, a certified Theta Healer, and a Yoga teacher (Sivananda in the Bahamas). He recently finished his autobiography.



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$$\frac{f_{9x-2}}{-11 \times 3} Q''$$

$$e = cc$$

$$y = 2x^2$$

$$y = \frac{dx}{dz}$$

$$\int (x \pm a)^4$$

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$$S = \int_1^{10} \frac{1}{x} dx$$

$$+ a^2$$

$$\frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\sin a = \frac{1}{b}$$

$$S = \int_1^{10} \frac{1}{x} dx$$

$$+ a^2$$

$$\frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$)- \dots$$

Seifert Surfaces and Genera of Knots

By Michael Landry

Assistant Professor, Department of Mathematics and Statistics, Saint Louis University

The subject of *topology* encompasses a huge swath of modern mathematics, and yet many undergraduates have never heard of it. One reason for this is that there is a relatively high barrier to entry in terms of definitions and background, so it tends to be taught in graduate and advanced undergraduate courses.

One of the most immediately accessible parts of topology is a field called *knot theory*, which is the study of circles tangled up in 3-dimensional space. Most of us have some experience tying knots (shoelaces, for example), and thus have some intuition that can help us when thinking about knot theory. I hope this short article piques your interest in topology!

If you want more details, I encourage you to look at Colin Adam's excellent textbook [1], which was the primary resource I used in preparing this.

1 Knots

Whatever the definition of a knot is, it should rigorously model the following situation: you have a string that is tangled up in some way, and you attach the 2 loose ends to each other. This could be by gluing them, taping them, or tying a knot, although the last method is linguistically problematic for obvious reasons. The result is a circular string that you can play with, and which we will call a *knot*. Maybe if you fiddle around with it for long enough, you can untangle it to a circle that lies flat on the table—no cutting allowed though—in this case, the knot you created is what we call the *unknot*. Otherwise, your string is knotted in some intrinsic sense. This knotting phenomenon is what knot theory explores.

If you do not like thinking of your mathematical objects being made out of string and tape, fine. Being more explicit, but still a little vague, think of a knot as an equivalence class of embeddings of the circle in three-dimensional space, where two embeddings are equivalent if one can be continuously deformed to the other such that each intermediate map is injective as

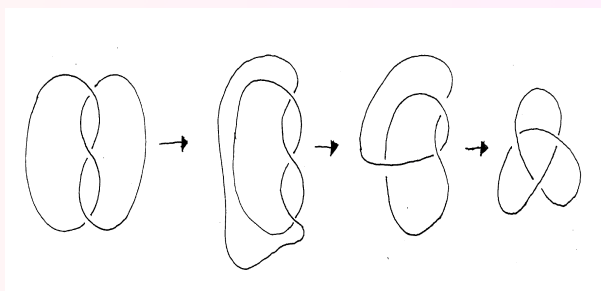


Figure 1: These are all the same knot, as the deformations above show.

in Figure 1.

2 Connected sums of knots

You're used to thinking about ways of combining two numbers to get another number (addition, multiplication, exponentiation, ...). We now define a way to combine two knots and get another knot! It's called *connected sum*. The connected sum of two knots, K_1 and K_2 , denoted $K_1 \# K_2$, is obtained in the following way: draw the two knots next to each other and remove a small segment from an outer arc of each knot. Now glue the two remaining parts together as in Figure 2.

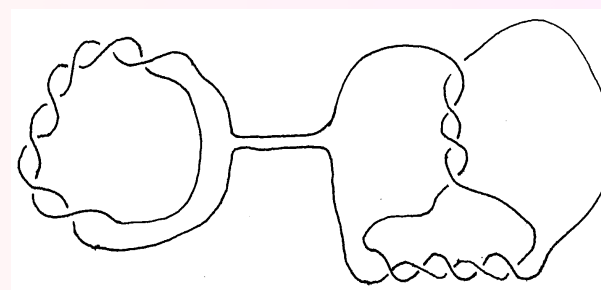
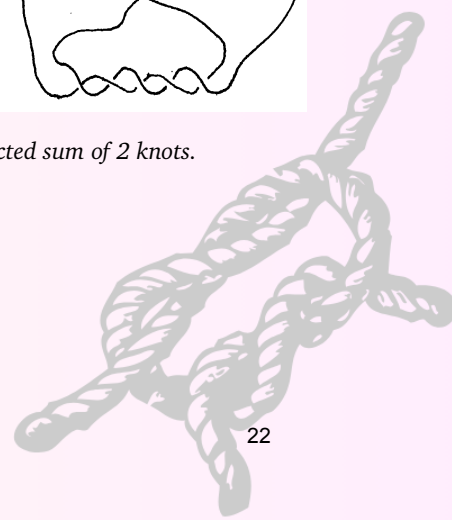


Figure 2: A connected sum of 2 knots.



It is not clear that this operation is well-defined, and, in fact, it is not unless we define it on *oriented* knots, where the gluing is determined by requiring that it preserves orientation. You might be a little nervous that we had to make a choice of which small segment to remove. This is not an issue as long as the segments we choose do not traverse any crossings in our drawings of the knots. Can you prove this?

What can we say about the connected sum ($\#$)? Here are some observations:

- (a) $\#$ is commutative.
- (b) $\#$ is associative.
- (c) if K is any knot and U is the unknot, then $K\#U = K$.

One next natural question to ask is whether $\#$ has inverses. That is, does there exist a knot K^{-1} such that $K\#K^{-1} = U$? Could a knot like the one shown in Figure 2 be the unknot? It turns out that the answer is no, and the rest of this article will be devoted to proving this result.

3 The classification of compact, connected, orientable surfaces

We need to make a few definitions.

A *surface* is a two-dimensional manifold. If you are not comfortable with that description, think of a surface as a space that looks like a two-dimensional plane if you zoom in really close to any point. A *surface with a boundary* is a space that looks like either a two-dimensional plane or the upper half of a two-dimensional plane if you zoom in on any point. The *boundary* of such a surface is composed of the “upper half plane” points. A surface is *connected* if any two points in the surface can be connected by a path that is also in the surface.

A surface is *compact* if it doesn’t “go off to infinity” anywhere. One way of phrasing this rigorously is to say S is compact if every continuous function $S \rightarrow \mathbb{R}$ has a maximum value somewhere in S . For example, the unit disk

$$D = \{(x, y) \in \mathbb{R}^2 \mid (x^2 + y^2 \leq 1)\}$$

is compact, but the punctured disk

$$D' = \{(x, y) \in \mathbb{R}^2 \mid 0 < (x^2 + y^2 \leq 1)\}$$

is not compact because the function $D' \rightarrow \mathbb{R}$ defined by $(x, y) \mapsto \frac{1}{x^2 + y^2}$ is unbounded. Alternatively, you can think of compact surfaces as the ones that can be constructed by gluing together finitely many triangles.

Define an *orientable surface* to be a surface (possibly with a boundary) that is 2-sided, in the sense that if a tiny person is standing on the surface at a point P and takes a walk that stays away from the boundary, it is impossible for them to return to P with their head pointing in the opposite direction.

Remark Orientability is usually defined in a different, more complicated way. But for surfaces embedded in an orientable 3-manifold, orientability and 2-sidedness are equivalent conditions. However, there are examples of three-dimensional manifolds that are nonorientable and admit two-sided embeddings of nonorientable surfaces (see [5], for example). Since we are picturing our surfaces as sitting in $\mathbb{R}^3$, which is orientable (note that I have not defined orientability for three-dimensional manifolds!), we are good to go.

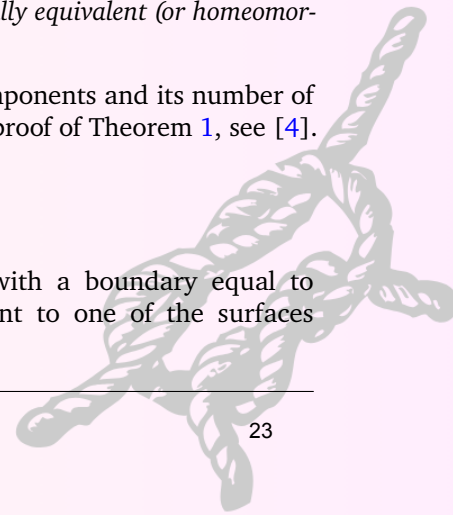
All those definitions pay off in the following classification theorem.

Theorem 1. *Every orientable, connected, compact surface with a boundary is topologically equivalent (or homeomorphic, if you know that word) to one of the surfaces shown in Figure 3.*

That is, any such surface is uniquely determined by its number of boundary components and its number of “holes,” which we call the *genus* of the surface (the plural of genus is *genera*). For a proof of Theorem 1, see [4].

4 Seifert surfaces

A Seifert surface for a knot K is a compact, orientable, connected surface with a boundary equal to K . By the previous section, any such surface must be topologically equivalent to one of the surfaces



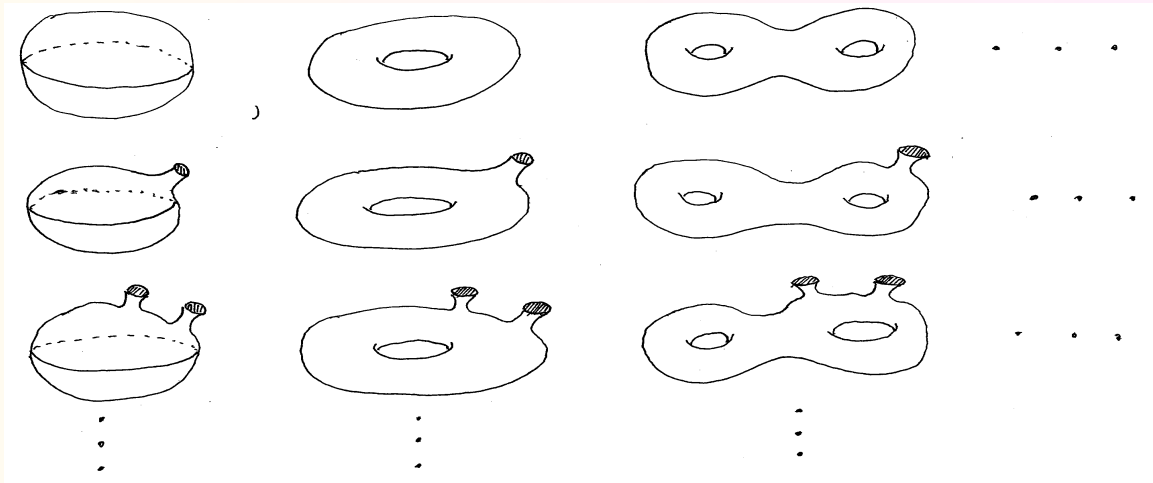


Figure 3: Classification of orientable, connected, compact surfaces with boundary.

in the second row of Figure 3 since it has only one boundary component. Therefore, any Seifert surface has a well-defined genus. The genus of a knot K , denoted $g(K)$, is the minimum genus of all Seifert surfaces for K .

Note that the definition above does not make sense unless we know that any knot possesses at least one Seifert surface. But this fact is not that hard to prove. In fact, Seifert himself gave a nice algorithm for producing a Seifert surface given a drawing of a knot, which is illustrated in Figure 4.

1. Pick an orientation for your knot.
2. Start tracing around the knot in your chosen direction. When you get to a crossing, turn right or left, in whichever direction agrees with the orientation of the knot. When you get back to where you started, start somewhere else in the drawing. Do this until you've traced over all the arcs in the drawing.
3. In the previous step, you created a collection of circles, called Seifert circles. Attach a disk to each Seifert circle, so that each Seifert circle now bounds a disk. In the case of concentric circles, vary the height of the disks so that the innermost Seifert circles are the highest.
4. At the site of each of the old crossings, attach the two neighboring disks by gluing in a rectangular band with a half twist, the direction of the twist being determined by the original crossing.

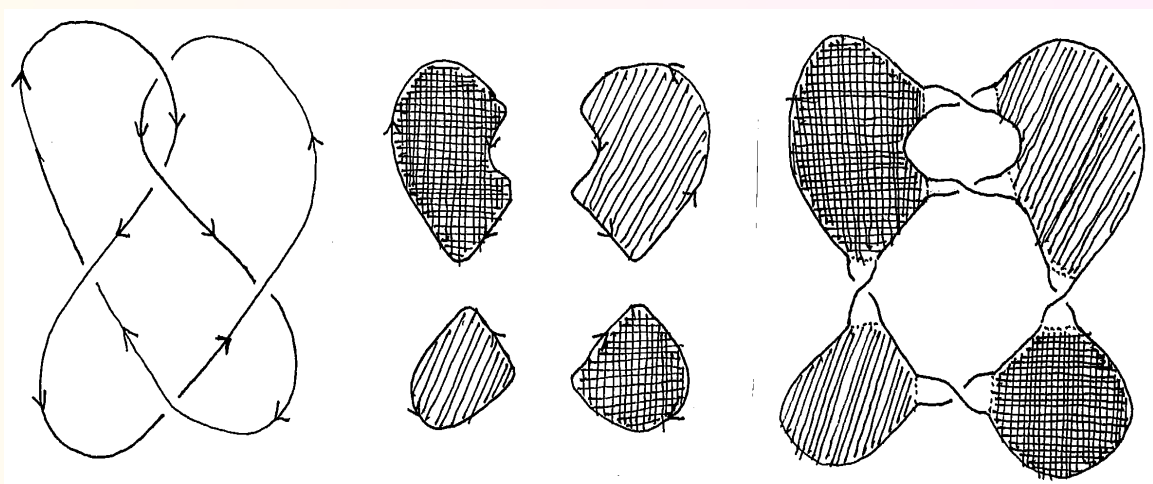


Figure 4

The resulting surface clearly has a boundary to the original knot. What is not immediately clear is that the resulting surface is orientable. But this is easy enough to see, and I leave it as an exercise for you, dear reader. Here's a hint: if the boundary of a disk has a counterclockwise orientation, paint its top red and its bottom blue.

Do the opposite for disks whose boundaries have a clockwise orientation. Why does the way we glued in the bands allow us to extend the red and blue paint over the entire surface such that the two colors never meet?

Example Consider the Seifert surface for the trefoil knot shown in Figure 5. By cutting and pasting as shown in the figure, we can see that this surface is homeomorphic to the torus minus a disk. You can also see this using the Euler characteristic, but cutting and pasting feels more fun to me.

This shows that the genus of the trefoil knot is less than or equal to 1. If the trefoil knot had genus 0, then it would bound a disk, but this is impossible (why?). Therefore, we have determined that the genus of the trefoil knot is 1.

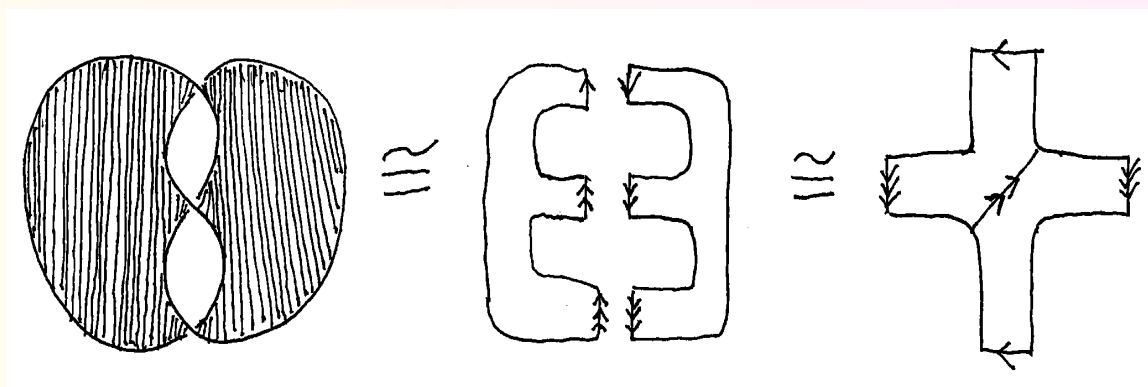


Figure 5: Left: A Seifert surface for the trefoil knot. Center: by cutting the lefthand surface open along a vertical plane intersecting the knot's crossings, we see that the surface is obtained by gluing together these two “3-shaped” surfaces in the way shown. Right: can you see why this surface is the same as a torus minus a disk?

Example Let's kick it up a notch. Consider the Seifert surface shown in Figure 6. Well, according to our definition, this is not strictly speaking a Seifert surface because its boundary is not a knot; it is a *link*. A link is just a collection of circles embedded in 3-space, up to the same equivalence relation that we used to define knots. We broaden our definition of the Seifert surface to include those whose boundaries are links. This particular link is called the *Borromean rings* and has the neat property that no two of the three circles are linked to each other, even though the link as a whole cannot be undone.

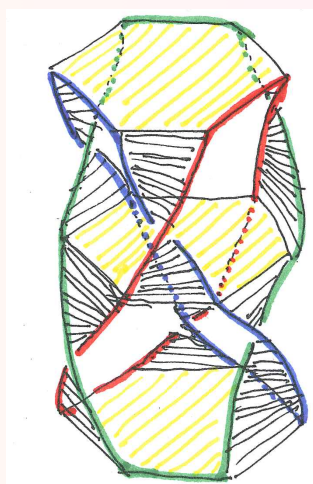


Figure 6: A Seifert surface for the Borromean rings.

We can obtain the surface in Figure 6 by gluing together three hexagons in the way prescribed by Figure 7, which shows that the surface is topologically equivalent to a torus minus three disks. Therefore the genus of the Borromean rings is less than or equal to 1.

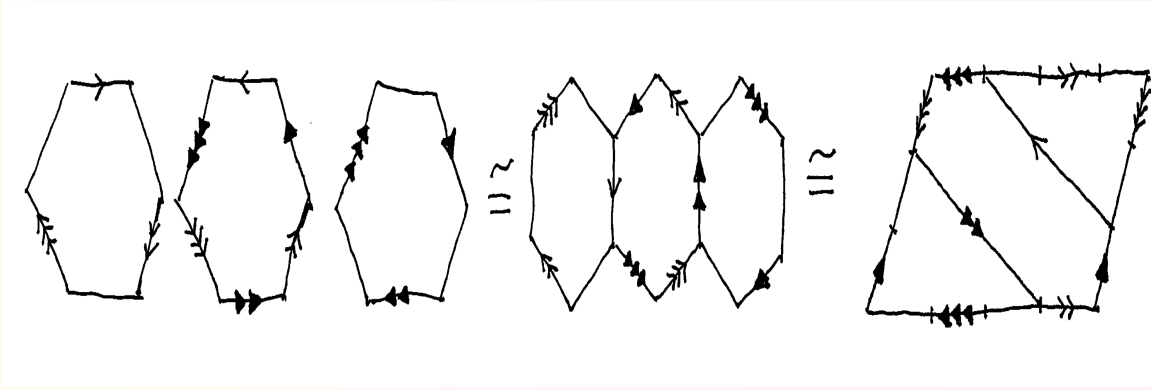


Figure 7

Note that “topologically equivalent” does not mean that we can continuously and injectively deform the surface in Figure 6 (à la our knot deformations) to look like the $(4, 2)$ entry of the table in Figure 3. Indeed, such a deformation would unlink the Borromean rings, which is impossible. We will not prove that - although it is not hard - the three component unlink has a property called *tricolorability*, which the Borromean rings do not have.

One last thing about this example—we have shown that the genus of the Borromean rings is at most one, but we have not shown that the Borromean rings do not bound a thrice-punctured sphere, so we cannot conclude that their genus is actually 1. The genus is in fact 1, but the only proof I am aware of uses the Thurston norm on homology, which falls beyond the scope of this article.

5 Main Theorem

It is time to finally get around to resolving that question about the “inverses” of knots from Section 2. Conveniently, this follows immediately from the following theorem describing the relationship between genus and connected sum.

Theorem 2. *Let K_1 and K_2 be knots. Then $g(K_1 \# K_2) = g(K_1) + g(K_2)$.*

Proof. First, observe that $g(K_1 \# K_2) \leq g(K_1) + g(K_2)$ because you can attach minimal genus Seifert surfaces for K_1 and K_2 by a rectangular band to get a Seifert surface for $K_1 \# K_2$ whose genus is the sum of the original two.

Now we prove that the reverse inequality holds. Take a Seifert surface Σ for $K_1 \# K_2$ with minimal genus. By assumption, there exists a sphere S , punctured twice by $K_1 \# K_2$, which separates K_1 from K_2 (see Figure 8).

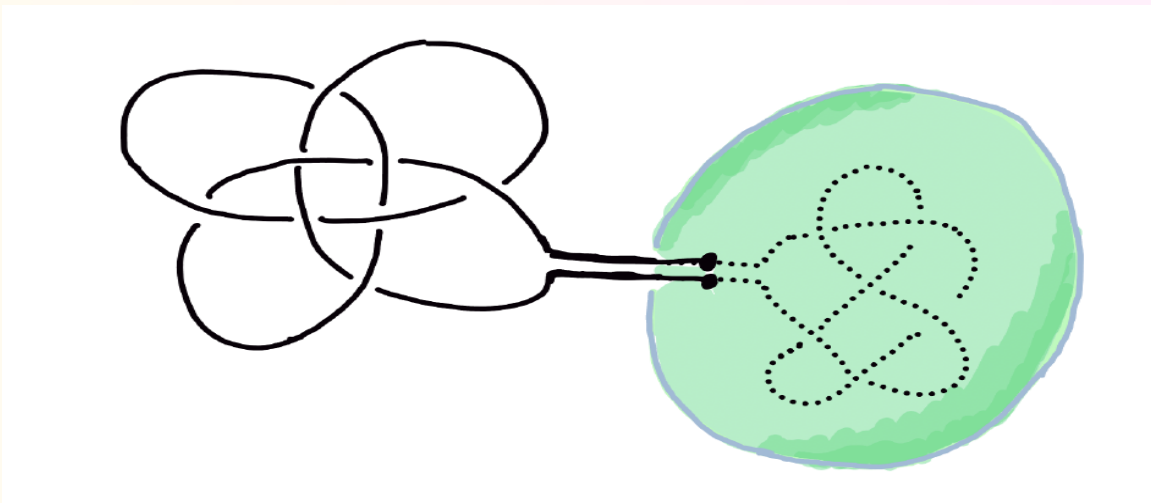


Figure 8: Given a connected sum $K_1 \# K_2$, there is always a sphere separating K_1 from K_2 and intersecting $K_1 \# K_2$ exactly twice. How might a Seifert surface for $K_1 \# K_2$ intersect this sphere?

By perturbing the sphere ever so slightly, we can preserve this separation property and guarantee that Σ intersects S in arcs, all but one of which are circles (the last one is a path P between the two punctures). The fact that we can do this follows from the theory of transversality, which you can read about in [3] if you are interested.

If there are no circles on the surface of S , we are done by cutting along P ; we do not change the genus, and we obtain Seifert surfaces for K_1 and K_2 , so $g(K_1 \# K_2) \geq g(K_1) + g(K_2)$.

Otherwise, there is an innermost circle on S , and by doubling S inside the circle, we create a new Seifert surface, which intersects S in one less circle.

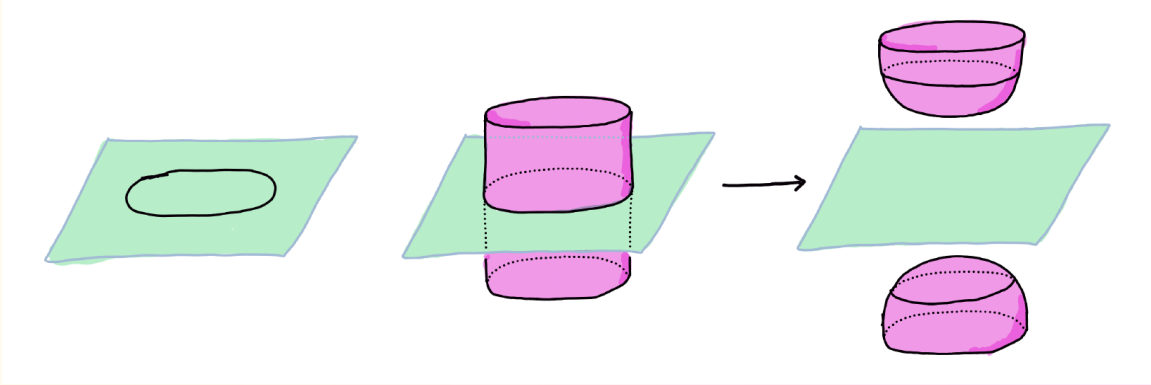


Figure 9: Our Seifert surface Σ intersects the sphere S (green) in one arc and some collection of circles. Look at the innermost circle (left). Near to the innermost circle, Σ (pink) is as shown in the center picture. We can cut Σ along the circle and cap it off as shown on the right to get a new surface that intersects S in one less circle.

If the resulting surface is disconnected, discard the component which does not connect to the knot. This surgery does not increase the genus, because it increases Euler characteristic by two without changing the number of boundary components. As an aside, this even shows that we always create a sphere component by this surgery since $g(\Sigma)$ is minimal.

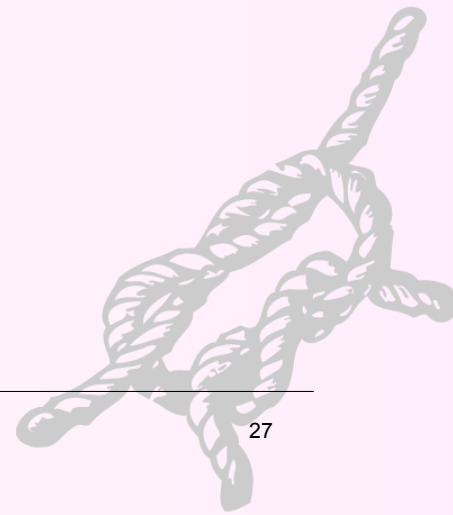
By repeating this type of surgery, we eventually reduce the number of intersecting circles to zero, so we are finished. □

Corollary 3. *If either K_1 or K_2 is nontrivial, then $K_1 \# K_2$ is not the unknot.*

Proof. As we remarked earlier, the only knot whose genus is zero is the unknot. □

References

- [1] Adams, Colin. *The knot book*. American Mathematical Society, 2001.
- [2] Candel, Alberto and Conlon, Lawrence. *Foliations II*. American Mathematical Society, 2003.
- [3] J. Lee. *Introduction to smooth manifolds*. Second edition. Springer, 2013.
- [4] Massey, William. *Algebraic topology: an introduction*. Harcourt, Brace & World, Inc. 1967.
- [5] Weeks, Jeffrey. *The shape of space*. Chapman & Hall, 2001.



What is a Zine?

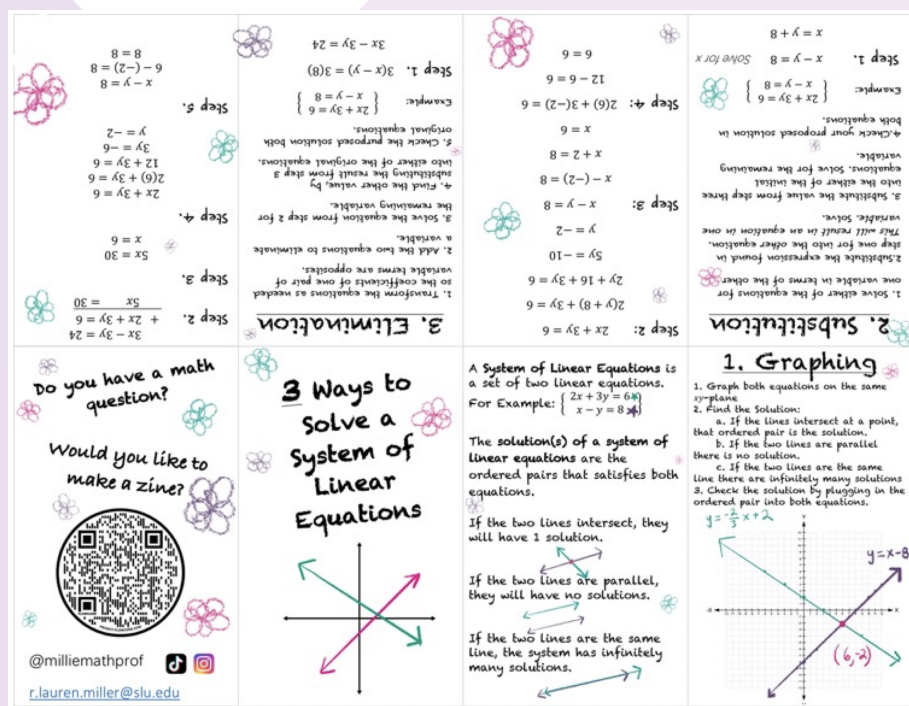
By Lauren Miller

Assistant Professor, Department of Mathematics and Statistics, Saint Louis University

When I finished my Master's in Mathematics at Saint Louis University (SLU), I wasn't sure what I wanted to do. I took a summer internship in accounting, I adjuncted, tried my hand at writing grants and fundraising, and found a home working in the Science and Technology department of the Saint Louis Public Library Central Branch. While there, I explored the 510's in the Dewey decimal system; this is where all the math books are located. We had books on math for children, teens, and adults. One place where there were no 510's was our Zine Collection. Started in 2015, the Zine Library was the first of its kind in St. Louis. A zine is a small-circulation, self-published work of original or appropriated texts and images, usually reproduced via a copy machine. The word "zine" is a shortened form of the term fanzine. Fanzines emerged as early as the 1930s among fans of science fiction. They also have roots in the informal, underground publications that focused on social and political activism in the '60s.

In 2016, I got my first zine, a zine about how to register and vote in St. Louis. Fast forward to 2020, and it is my first semester teaching as a faculty member at SLU. We were still in the thick of the COVID-19 pandemic. I was trying to find a way to engage with my students in a time when we still felt very disconnected; six feet apart and with half of our faces covered by masks.

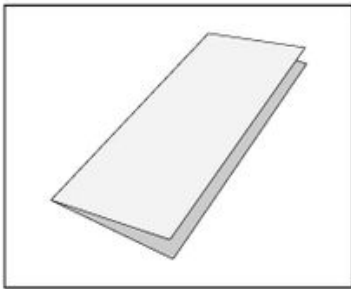
I thought back to my time at the SLPL and decided to use zines, specifically mini-zines, as a teaching and learning tool. Mini-Zines are zines that can be made from one 8.5 by 11 sheet of paper, folded, and cut once to have eight pages. Since then, I have made zines for my Intermediate and College Algebra courses, as well as asked students to make their own zines in my Mathematics Through Literature Ignite Seminar. Below is an example of a zine I made about solving systems of equations.



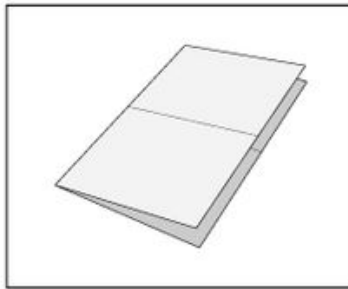
In October of 2023, SLU's Association for Women in Mathematics (AWM) invited me to speak, and I agreed before I had any idea of what I would talk about. So I talked about Zines and how I use them in my classroom. We then, as a group, created a Zine about the AWM to use as a tool for recruitment that we can share with students across campus. On the opposite page is the zine created by Wedad Alharbi, Sydney Bement, Olivia Frank, Dorsa Ghoreishi, Stacey Harris, Dhananjani Welagedara, and me. The instructions on how to fold it can be found below. I hope you will take the time to make the zine and share it with others!

How to Make a Mini Booklet from a Sheet of Paper

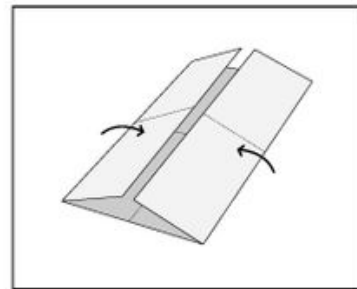
1. Fold a sheet of paper in half lengthwise.



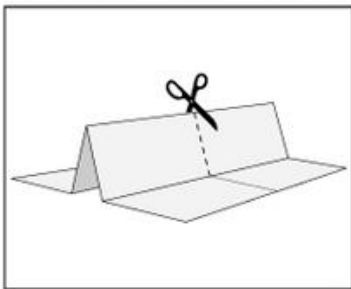
2. Unfold, then fold the paper in half widthwise.



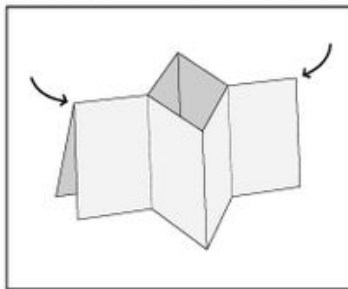
3. Unfold, then fold the edges up to the center fold.



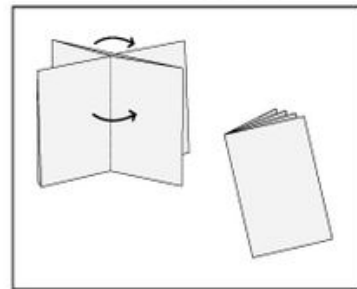
4. Unfold, then fold widthwise and cut a slit only half way down the center.



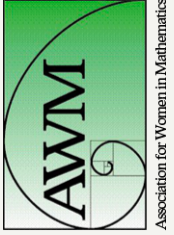
5. Unfold, then fold lengthwise and push the ends in towards the center.



6. Fold all the pages together into a mini booklet.



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Word Bank:

- Lovelace
- Germain
- Noether
- Hypatia
- Uhlenbeck
- Jackson
- Johnson
- Vaughan

Word Search

DE RHE T E O N
G E R M A I N O X S
N W E C A L E V O L
K C E B N E L H U K
D W A H Y P A T I A
A O D D C R N Q E
T N G U A V I J
J A K S O N X Y Z
V J X T M G O R J
L O N S N H O J

- The Marriage Problem
- Pi Day
- Speaker Series
- Topological Tic Tac Toe

Upcoming Events



Website:

Past Events



Why AWM?

There is a place for everyone, no matter gender nor major! The purpose of the Association for Women in Mathematics is to create a community in which women and girls can thrive in their mathematical endeavors, and to promote equitable opportunity and treatment of women and others of marginalized genders and gender identities across the mathematical sciences.

Mission Statement

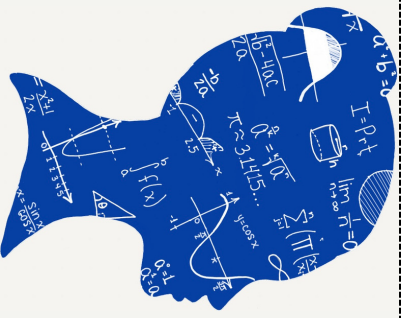
Participate in weekly events designed to encourage women and other minorities in the mathematical sciences.

Contribute to the community of undergraduate students, graduate students, and faculty in support of women and other gender minorities.



SAINT LOUIS UNIVERSITY STUDENT CHAPTER:

ASSOCIATION FOR WOMEN IN MATHEMATICS



Meeting Information

When: Mondays 4pm

**Where: Ritter Hall,
Room 323**

Contact Us!

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